

# All of Math in Three Pages

## INDIVIDUAL TEST TOPICS

Here is an overview of “high school mathematics.” Most of the non-Calculus topics that you have studied and a lot that you haven’t will appear somewhere in the following outline. In some cases a formula or example is given, but many of these important results are just named, which leaves the interesting task of discovering how they work for you to unearth during coaching sessions or World History (just kidding). Caution: do not attempt to absorb all of this material in one sitting! Many of these tidbits, such as the Principle of Inclusion-Exclusion, require more than a cursory amount of practice to understand. However, with these tools under your belt and a healthy dose of ingenuity you should be ready to tackle any problem. Especially those in *The Mandelbrot Problem Book*, a great source of practice problems with hints and answers, available through Greater Testing Concepts at <http://www.mandelbrot.org/resources.html>.

### I. Arithmetic

- A. Number Systems: natural numbers, integers, rationals/irrationals, reals
- B. Order of operation, properties of real numbers, scientific notation
- C. Common sums:  $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$ , first  $n$  squares, or first  $n$  cubes
- D. Complex numbers
  - 1. Addition, subtraction, multiplication, and division
  - 2. Conjugate, absolute value, real/imaginary parts
  - 3. Polar form of complex numbers and De Moivre’s Theorem

### II. Algebra

- A. Standard algebraic tasks
  - 1. Adding, subtracting, multiplying, and dividing algebraic expressions
  - 2. Linear functions (slope, equations, and graphs)
  - 3. Solving equations and word problems
  - 4. Working with absolute value
  - 5. Graphing equations and inequalities
- B. Polynomials
  - 1. Finding roots (quadratic equation, rational root theorem, complex roots)
  - 2. Factoring (trinomials,  $x^n - y^n$ ,  $x^n + y^n$ )
  - 3. Relationship of roots to coefficients, discriminant, Newton’s sums
  - 4. Finite differences, interpolation
- C. Radicals
  - 1. Simplifying expressions involving radicals
  - 2. Solving equations with radicals
- D. Proportions, direct and inverse variation
- E. Functions
  - 1. Domain, range, operations with functions, composition
  - 2. One-to-one, inverse functions
  - 3. Several variables, such as  $f(x, y, z)$
  - 3. Functional equations, i.e.  $f(x + y) = f(x)f(y)$
- F. Exponents and Logarithms
  - 1. Definitions, meaning of negative or fractional exponents
  - 2. Laws, such as  $a^m/a^n = a^{m-n}$ ,  $\log_b a^m = m \log_b a$ , or  $\log_b a = \log_c a / \log_c b$
  - 3. Solving equations with exponents and logarithms
- G. Vectors
  - 1. Adding, subtracting, multiplying by a scalar
  - 2. Dot product, cross product (geometric interpretation)

### III. Geometry

#### A. Fundamentals

1. Points, lines, planes, distance, intersection
2. Acute, right, obtuse, vertical, complementary, supplementary angles
3. Perpendicular/parallel lines, theorems on transversals
4. Triangle, isosceles, equilateral, bisectors
5. Quadrilateral, rectangle, square, parallelogram, rhombus, trapezoid

#### B. Triangle terminology and notation

1. Side lengths of  $\triangle ABC$  are usually  $BC = a$ ,  $AC = b$ , and  $AB = c$
2. Angles of  $\triangle ABC$  are often  $m\angle A = \alpha$ ,  $m\angle B = \beta$ , and  $m\angle C = \gamma$
3. Perimeter is  $p$ , semiperimeter is  $s = \frac{1}{2}p$ , altitude is  $h$
4. Inradius is  $r$ , circumradius is  $R$ , area is  $K$
5. Centroid is  $G$ , incenter is  $I$ , circumcenter is  $O$ , orthocenter is  $H$

#### C. Triangles

1. Conditions for congruence/similarity, proportions
2. Area formulas:  $K = \frac{1}{2}bh$ ,  $K = \frac{1}{2}ab\sin C$ ,  $K = rs$ ,  $K = abc/4R$
3. Hero's formula  $K = \sqrt{s(s-a)(s-b)(s-c)}$
4. How to find lengths of inradius, circumradius, altitude, median, or angle bisector
5. Ratio in which  $G$ ,  $H$ , and  $I$  divide median, altitude, and angle bisector
6. Standard results: Pythagorean Theorem, Angle Bisector Theorem
7. Classic results: Ceva's Theorem, Stewart's Theorem, Nine-Point Circle

#### D. Circles

1. Radius, diameter, circumference, area, sector, arc, chord, tangent, secant
2. Relationships between central angles, inscribed angles, and intercepted arcs
3. Power of a point theorem for chords, secants, and tangents
4. Cyclic quadrilaterals
  - a. Requirements which ensure that four points lie on a circle
  - b. Ptolemy's Theorem, Brahmagupta's formula for area

#### E. Transformations: rotations, translations, reflections, and dilations

#### F. Polygons: perimeter, area, angle measures, regular polygons

#### G. Analytic geometry

1. Midpoint/distance formulas, slopes of parallel or perpendicular lines
2. Conic sections: parabolas, circles, ellipses, hyperbolas
  - a. Terminology (focus, directrix, axes, vertices, center, asymptotes)
  - b. Relationship between equations and graphs
  - c. Geometry of conic sections
3. Finding points of intersection of curves, conditions for tangency
4. Lattice points, area of polygon given coordinates, Pick's theorem

#### H. Solid geometry

1. Parallel/skew lines, parallel/perpendicular planes, dihedral angles
2. Formulas for boxes, spheres, pyramids, cones, cylinders, and frustums

### IV. Trigonometry

#### A. Degrees/radians, quadrants, definition of trig functions via the unit circle

#### B. Finding values of other trig functions of a given angle from one of them

#### C. Sine, cosine, and tangent in a right triangle, use in word problems

#### D. Law of Cosines, Extended Law of Sines, solving general triangles

#### E. Identities

1. Basic:  $\sin^2 x + \cos^2 x = 1$ ,  $\cot x / \cos x = \csc x$
2. Double angle formulas, i.e.  $\cos 2x = 1 - 2\sin^2 x$
3. Half angle formulas, such as  $\tan\left(\frac{x}{2}\right) = (1 - \cos x) / \sin x$
4. Angle addition/subtraction formulas for sine, cosine, and tangent
5. Special values, such as  $\sin 15^\circ = \frac{1}{4}(\sqrt{6} - \sqrt{2})$

F. Solving trig equations, for instance  $\cos(3\theta + 30^\circ) = \frac{\sqrt{3}}{2}$

G. Graphs of trig functions (period, frequency, amplitude)

H. Inverse trig functions

I. Polar coordinates, equations

## V. Number Theory

A. Divisibility, greatest common factor, least common multiple

B. Primes, composites, relatively prime integers, perfect numbers

C. Formulas involving divisors of  $n$

1.  $d(n)$ , the number of divisors of  $n$

2.  $\sigma(n)$ , the sum of the divisors of  $n$

3.  $\phi(n)$ , the number of positive integers less than  $n$  relatively prime to  $n$

D. Congruences (modular arithmetic)

1. Laws for working with congruences (+, -,  $\times$ , and  $\div$ )

2. How to find the remainder when  $a^m$  is divided by  $n$

3. Fermat's Little Theorem ( $a^{p-1} \equiv 1 \pmod{p}$ ) and Euler's extension

E. Different number bases (such as binary)

F. Continued fractions

G. Linear Diophantine equations and Pell equations

H. Pythagorean triples, generated by formulas  $a = u^2 - v^2$ ,  $b = 2uv$ ,  $c = u^2 + v^2$

I. Facts about perfect squares (remainders mod 4, final digits, etc.)

## VI. Probability and Combinatorics

A. Counting

1. Multiplication principle, factorials:  $n! = n(n-1) \cdots (2)(1)$

2. Permutations and combinations,  $\binom{n}{k}$  notation

3. Principle of Inclusion-Exclusion

4. Partitions, generating functions

B. Binomial coefficients, expansions, and elementary identities

C. Computing probability

1. Addition and multiplication principles

2. Common tasks (i.e. probability of drawing two hearts from a standard deck)

3. Geometric methods, using ratios of areas or volumes

D. Expected value

## VII. Sequences and Series

A. Recursive definition for sequences, sigma/product notation

B. Arithmetic and geometric sequences

1. Finding  $n^{\text{th}}$  term and sum of  $n$  terms

2. Sum of infinite geometric series:  $S = \frac{a}{1-r}$

C. Telescoping sums

D. General solution to linear recursion  $a_{n+2} = Aa_{n+1} + Ba_n$

## VIII. Everything Else

A. Inequalities

1. Min or max of a set

2. Triangle inequality

3. Arithmetic-Geometric mean:  $\frac{1}{n}(x_1 + x_2 + \cdots + x_n) \geq \sqrt[n]{x_1 x_2 \cdots x_n}$

4. Cauchy-Schwarz, power means

5. Weighted inequalities, equality conditions

B. Pigeon-hole principle

C. Logic

1. Boolean operators (AND, OR, NOT, IF/THEN)

2. Systems of consistent statements