Team Play Solutions

Part i: Proving that $\triangle ISU \cong \triangle IWU$ is fairly routine—we are given that $\overline{US} \cong \overline{UW}$, we know that $\overline{IS} \cong \overline{IW}$ since each segment is a radius of the incircle, and clearly $\overline{IU} \cong \overline{IU}$. So $\triangle ISU \cong \triangle IWU$ by SSS.

But it is also possible to prove that $\triangle ISU \cong \triangle IWU$ without using the fact that $\overline{US} \cong \overline{UW}$. Just argue that $\overline{IS} \cong \overline{IW}$ and $\overline{IU} \cong \overline{IU}$ as before, observe that $\angle IWU$ and $\angle ISU$ are right angles because W and S are points of tangency, then use HL (hypotenuse-leg) congruence. (This is how it is possible to deduce that $\overline{US} \cong \overline{UW}$ in the first place.)

Regardless, we next show that $m\angle WIS = m\angle B$. Since the sum of the angles of quadrilateral $IWUS$ is 360° and $\angle IWU$ and $\angle ISU$ are right angles, it follows that $m\angle WIS + m\angle WUS = 180°$. But we also have $m\angle AUV + m\angle WUS = 180°$, so we deduce that $m\angle WIS = m\angle AUV$. Lastly, $m\angle AUV = m\angle ABC = m\angle B$, by the given.

Dividing the above result by two gives $\frac{1}{2}m\angle WIS = \frac{1}{2}m\angle B$. Since $m\angle WIU = m\angle SIU$ and together these two angles make up $\angle WIS$, we conclude that each is equal to $\frac{1}{2}m\angle WIS$. Hence $m\angle SIU = \frac{1}{2}m\angle B$.

Part ii: It is common knowledge that the three angle bisectors of a triangle each pass through the center I of the inscribed circle. Therefore we know that $m\angle RBI = \frac{1}{2}m\angle B$, just as $m\angle SIU = \frac{1}{2}m\angle B$. (If desired, one could prove that $\triangle RBI \cong \triangle TBI$ using HL congruence, but it is OK to omit this step.) Furthermore, $m\angle BRI = m\angle ISU = 90°$. Therefore $\triangle BRI \sim \triangle ISU$ by AA similarity.

Comparing ratios in these similar triangles yields $SU/IS = RI/BR$. But $IS = IR = r$ and $BR = s_b$ by the givens. Therefore multiplying through by IS yields $SU = r^2/s_b$, as desired.

Part iii: In an analogous fashion one may show that $\triangle ITV \sim \triangle CRI$, which leads to $TV = r^2/s_c$ upon comparing equal ratios. (We omit the details.) Next note that $UV = UW + WV = SU + TV$. Thus

$$UV = \frac{r^2}{s_b} + \frac{r^2}{s_c} = r^2\left(\frac{s_b + s_c}{s_b s_c}\right) = r^2\left(\frac{a}{rr_a}\right) = \frac{ar}{r_a},$$

where we used the fact that $s_b + s_c = 2s - b - c = (a+b+c) - b - c = a$ and also that $s_b s_c = rr_a$, as mentioned in the Facts section.

Part iv: It is possible to prove the given equality solely by algebraic means, utilizing some of the relationships listed in the Facts section. It is quite satisfying to watch this identity resolve itself, if you enjoy that sort of thing. We leave the demonstration to the interested reader, and instead follow the advice to try a geometric approach.

Based on our previous work, we recognize the quantities appearing in the given expression. Thus $s_a - (r^2/s_c) = AT - TV = AV$, and in the same way $s_a - (r^2/s_b) = AS - SU = AU$. Of course $b = AC$ and $c = AB$, so proving the given equality is tantamount to proving that $AC/AB = AV/AU$. And clearly this would follow at once if we could show that $\triangle ABC \sim \triangle AUV$. But this is straight-forward; we are given that $\angle AUV \cong \angle ABC$ and both triangles share $\angle A$. Hence the triangles are similar by AA, and you're done.

Part v: To prove this unexpected area equality we will first find an expression for *area*$(BCUV)$. To begin, since $\triangle ABC \sim \triangle AUV$ we know that the ratio of their areas is equal to the square of the ratios of their sides. Thus

$$\frac{\textit{area}(AUV)}{\textit{area}(ABC)} = \left(\frac{UV}{BC}\right)^2 = \left(\frac{ar/r_a}{a}\right)^2 = \frac{r^2}{r_a^2}.$$

Letting $K = \textit{area}(ABC)$, we have $\textit{area}(AUV) = Kr^2/r_a^2$, which means that $\textit{area}(BCUV) = \textit{area}(ABC) - \textit{area}(AUV) = K - Kr^2/r_a^2$. Hence

$$\textit{area}(BCUV) = K\left(1 - \frac{r^2}{r_a^2}\right) = r_a s_a\left(1 + \frac{r}{r_a}\right)\left(1 - \frac{r}{r_a}\right).$$

Multiplying $r_a(1 + r/r_a)$ gives $(r + r_a)$, which is one part of the desired expression. It remains to show that $s_a(1 - r/r_a) = UV = ar/r_a$. This is equivalent to $s_a(r_a - r) = ar$, upon multiplying through by r_a. Using the fact $r_a s_a = rs$ we find that

$$s_a(r_a - r) = r_a s_a - s_a r = rs - (s-a)r = ar,$$

which completes the argument.

Part vi: This formula is reminiscent of Hero's formula $K = \sqrt{s s_a s_b s_c}$ for a triangle. Except here we have a quadrilateral, and we are multiplying sides lengths instead of quantities like s_a. Nonetheless, the formula is correct, as we shall show.

First note that $BV = BT + TV = s_b + r^2/s_c = (s_b s_c + r^2)/s_c$. In the same manner we find that $CU = (s_b s_c + r^2)/s_b$. Therefore we have

$$\begin{aligned}
\sqrt{(BC)(BV)(CU)(UV)} &= \sqrt{a\left(\frac{s_b s_c + r^2}{s_c}\right)\left(\frac{s_b s_c + r^2}{s_b}\right)\left(\frac{ra}{r_a}\right)} \\
&= a(s_b s_c + r^2)\sqrt{\frac{r}{(s_b s_c) r_a}} \\
&= a(r r_a + r^2)\sqrt{\frac{r}{(r r_a) r_a}} \\
&= \left(\frac{ra}{r_a}\right)(r_a + r) \\
&= (UV)(r_a + r) \\
&= \mathit{area}(BCUV),
\end{aligned}$$

according to the result of the previous part.

Well-read students may be familiar with a delightful area formula for quadrilaterals which possess both an incircle and a circumcircle (neither condition occurs in general). It states that if the quadrilateral has sides of length a, b, c and d then its area is given by $\sqrt{abcd}$. This provides an alternate means of proving the given statement, as long as one explains why this formula works, and also verifies that $BCUV$ has a circumcircle.

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The Mandelbrot Team Play

Round Three Solutions

Greater Testing Concepts
PO Box 760
Potsdam, NY 13676

The Mandelbrot Team Play
web.mandelbrot.org
info@mandelbrot.org