Team Play Topics

ROUND TWO

The first section of the Round Two Mandelbrot Team Play is reproduced below. A list of topics and practice problems are also provided to aid in preparation. Note that these problems are not meant to serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Facts: The Fibonacci numbers are the sequence of integers 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... in which each number is the sum of the two previous numbers. The k^{th} Fibonacci number is written F_k, so this sequence satisfies $F_0 = 0$, $F_1 = 1$, and $F_{k+1} = F_k + F_{k-1}$ for $k \geq 1$.

TOPICS: Fibonacci numbers, recursion, induction, binomial coefficients, Pascal's triangle

Practice Problems

1. Show that $F_{n+1} = 2F_{n-1} + F_{n-2}$ for all $n \geq 2$.

2. Prove that $F_n^2 + F_{n+1}^2 = F_{n-1}F_{n+1} + F_nF_{n+2}$ for all $n \geq 1$.

3. Recall that $k!$ means the product of all positive integers from k down to 1. In other words, $k! = (k)(k-1)\cdots(2)(1)$. For instance, $3! = 6$ and $5! = 120$. Make a conjecture as to the value of the sum $n(n!) + (n-1)(n-1)! + \cdots + 2(2!) + 1(1!)$. Then prove your conjecture by induction.

4. For $n \geq k \geq 0$ the binomial coefficient $\binom{n}{k}$ is given by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Use this formula to compute $\binom{9}{4}$ and $\binom{5}{0}$. Review the connection between binomial coefficients and Pascal's triangle.

5. It is not immediately apparent from the definition of $\binom{n}{k}$ that this quantity is an integer, rather than a fraction. To explain why this is so, first prove that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ by using the formula. How does this demonstrate that binomial coefficients are always integers?

Hints and answers on the next page. $\Longrightarrow$

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Hints and Answers

1. Rewrite $2F_{n-1} + F_{n-2}$ as $(F_{n-1} + F_{n-2}) + F_{n-1}$. The first two terms combine to give F_n, then adding F_{n-1} gives a grand total of F_{n+1}, as desired.

2. Rearrange the given identity to obtain $F_{n+1}^2 - F_{n-1}F_{n+1} = F_nF_{n+2} - F_n^2$. Then factor out F_{n+1} from the left-hand side and F_n from the right-hand side and simplify the resulting expressions.

3. We find that $4(4!) + 3(3!) + 2(2!) + 1(1!) = 119$, which is almost equal to $5! = 120$. Hence we conjecture that in general the overall sum is $(n+1)! - 1$. To prove this by induction, we first check the base case $n = 1$. Indeed, $1(1!) = 2! - 1$. Now suppose that the formula works for a particular value of n, say $n = k$. We wish to check that the formula also works for the next value, which is $n = k + 1$. We find that

$$\begin{aligned}(k+1)(k+1!) + k(k!) + \cdots + 2(2!) + 1(1!) &= (k+1)(k+1!) + [(k+1)! - 1] \\ &= [(k+1)+1](k+1)! - 1 \\ &= (k+2)(k+1)! - 1 \\ &= (k+2)! - 1,\end{aligned}$$

which is the formula for the case $n = k + 1$. This completes the proof.

4. We compute $\binom{9}{4} = \frac{(9)(8)(7)(6)(5)(4)(3)(2)(1)}{(4)(3)(2)(1)(5)(4)(3)(2)(1)} = \frac{(9)(8)(7)(6)}{(4)(3)(2)(1)} = (3)(7)(6) = 126$. In a similar manner, $\binom{5}{0} = 1$, using the fact that $0! = 1$. The n^{th} row of Pascal's triangle simply lists the binomial coefficients from $\binom{n}{0}$ up to $\binom{n}{n}$.

5. We can combine the expressions for $\binom{n}{k}$ and $\binom{n}{k+1}$ by first finding a common denominator. We find that

$$\begin{aligned}\binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} \\ &= \frac{(k+1)n!}{(k+1)k!(n-k)!} + \frac{(n-k)n!}{(n-k)(k+1)!(n-k-1)!} \\ &= \frac{(k+1)n!}{(k+1)!(n-k)!} + \frac{(n-k)n!}{(k+1)!(n-k)!} \\ &= \frac{(n+1)n!}{(k+1)!(n-k)!} \\ &= \frac{(n+1)!}{(k+1)!(n-k)!} = \binom{n+1}{k+1}.\end{aligned}$$

This says that each entry in the $(n+1)^{\text{st}}$ row of Pascal's triangle is equal to the sum of two entries from the n^{th} row. Hence if all the numbers in some row are integers, then all the numbers in the next row will also be integers. But it is is clear that the first row is composed of integers, and hence so are all subsequent rows, by induction.