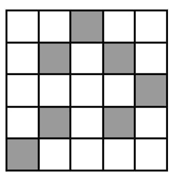
Answer Key

1. 7
2. $\frac{7}{25}$
3. 720
4. $\frac{3}{5}$
5. $1/(-x+2)$
6. 5051
7. $i\sqrt{3}$

1. Each row must contain at least one shaded square, but adjacent rows have different numbers of shaded squares, which means that there will need to be at least two more shaded squares, in the second and fourth row. The question is whether seven shaded squares can be arranged so that the columns also satisfy the given conditions. As the diagram demonstrates, this can be done, so **7** shaded squares suffice.



2. Although it is usually not advantageous to create fractions when solving an equation, we will make an exception here and divide through by 5 before squaring both sides. The result is

$$(\sqrt{1+x})^2 + 2(\sqrt{1+x})(\sqrt{1-x}) + (\sqrt{1-x})^2 = \left(\frac{7\sqrt{2}}{5}\right)^2.$$

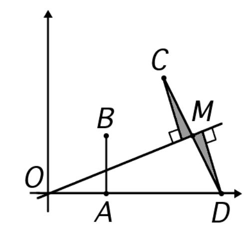
Simplifying, we obtain

$$\begin{aligned} (1+x) + 2\sqrt{1-x^2} + (1-x) &= \frac{98}{25} \\ \Rightarrow \quad 2 + 2\sqrt{1-x^2} &= \frac{98}{25} \\ \Rightarrow \quad \sqrt{1-x^2} &= \frac{24}{25} \\ \Rightarrow \quad x^2 = 1 - \frac{576}{625} &= \frac{49}{625}. \end{aligned}$$

Therefore the positive solution to this equation is **7/25**.

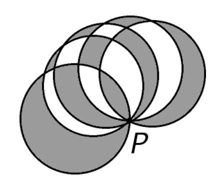
3. Suppose that Austin currently possesses x shirts, y pairs of pants, and z pairs of shoes. If he were to obtain one more shirt, then he would gain yz new outfits, since each would involve the new shirt, one of y pairs of pants, and one of z pairs of shoes. Similarly, buying another pair of pants leads to xz additional outfits, and more shoes gives xy extra outfits. We know that $yz = 48$, $xz = 90$, and $xy = 120$. Multiplying these and taking the square root gives $\sqrt{x^2y^2z^2} = \sqrt{(48)(90)(120)}$, which reduces to $xyz = 720$. In other words, Austin currently can create **720** outfits.

4. The key to solving the problem lies in determining the line through O equidistant from C and D. It turns out that this line is the one passing through the midpoint M of segment $\overline{CD}$. Note that $\overline{CM}$ and $\overline{DM}$ are not themselves perpendicular to OM. However, if we draw the segments that are perpendicular, creating the slim shaded right triangles, then the resulting triangles are congruent. (Proof?) This shows that line OM is the same distance from C and D. Hence any point P on $\overline{AB}$ above the intersection with OM will result in a line that is closer to C than to D. We find that M has coordinates $(1, \frac{5}{2})$, so line OM has equation $y = \frac{2}{5}x$, which intersects the line $x = 1$ at the point $(1, \frac{2}{5})$. Thus P can be located anywhere on the upper portion of $\overline{AB}$, which has length **3/5**.



5. One means for deducing the correct function $g(x)$ is to observe that if a and b are any two consecutive terms in the sequence, then $a + 1/b = 2$. (Try this with a few pairs of terms to see how this works.) Solving for b leads to $b = 1/(2-a)$. Therefore the desired function is given by $g(x) = 1/(2-x)$, or $g(x) = 1/(-x+2)$. (Either answer is fine.)

6. We count how many regions are formed by n circles for several small values of n in order to understand what happens as we add each successive circle. (The case $n = 5$ is shown at right, in which 16 regions are formed.) Tabulating our results for $n = 1$ to 5 yields



$n =$	0	1	2	3	4	5
# regions	1	2	4	7	11	16

Notice that subtracting adjacent values along the bottom row of the table gives 1, 2, 3, 4, and 5. This makes sense, because in general the n^{th} circle will cut through n existing regions, creating n more regions than before. Hence 100 circles will form

$$1 + (1 + 2 + 3 + 4 + \cdots + 100) = 1 + \tfrac{1}{2}(100 \cdot 101) = \mathbf{5051} \text{ regions.}$$

7. Using absolute value notation to denote distance, we are given

$$|\alpha^2 - 1| = 2|\alpha - 1|, \qquad |\alpha^4 - 1| = 4|\alpha - 1| = 2|\alpha^2 - 1|.$$

The final equality follows by substituting in the first equality. But we can rewrite $|\alpha^2 - 1|$ as $|(\alpha+1)(\alpha-1)| = |(\alpha+1)| \cdot |(\alpha-1)|$, so the first equality simplifies to just $|\alpha + 1| = 2$. (We can safely cancel $|(\alpha - 1)|$ since $\alpha \neq 1$.) Similarly, factoring $\alpha^4 - 1$ as $(\alpha^2+1)(\alpha^2-1)$ allows us to rewrite the second equality as $|\alpha^2 + 1| = 2$. At this point we substitute $\alpha = a + bi$ and use the fact that $\alpha^2 = (a^2 - b^2) + 2abi$ and that $|a + bi|^2 = a^2 + b^2$ to deduce that

$$(a+1)^2 + b^2 = 4, \qquad (a^2 - b^2 + 1)^2 + 4a^2b^2 = 4.$$

The first equation gives $b^2 = -a^2 - 2a + 3$, which can then be substituted into the second equation. The resulting equality miraculously reduces all the way down to $a^2 - a = 0$, so $a = 0$ or $a = 1$. These solutions in turn yield $b = \pm\sqrt{3}$ and $b = 0$, respectively. Since we must have $b > 0$, the unique number α satisfying the problem is $\alpha = i\sqrt{\mathbf{3}}$.

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⋆ National Level ⋆

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