Answer Key

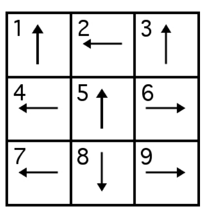
1. $\frac{10}{11}$	4. 6
2. 7	5. $\frac{10}{3}$
3. -3	6. $\frac{1}{7}(2\sqrt{2}-8)$
	7. 8

1. We can regroup and rewrite the sum on top as

$$(1+19)+(2+18)+\cdots+10=(10+10)+(10+10)+\cdots+10.$$

Since there were 19 terms in the top sum, its value is $19\cdot 10$. By the same reasoning, the bottom sum is $19\cdot 11$. Hence the ratio reduces to just $\mathbf{\frac{10}{11}}$. Note that an answer of $\frac{190}{209}$ should *not* receive credit.

2. The most convincing solution to this problem is to cut out nine small squares, draw an arrow on each, then set up the given arrow maze and follow the directions. After wandering around for a surprisingly long time within the little 3×3 grid, you should discover yourself retracing your initial path through the maze in reverse, finally exiting at square **7**. The final position of the arrows is shown at left, so that you can check yourself.



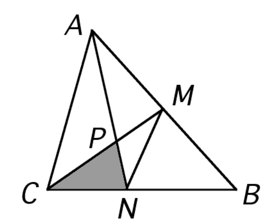
3. After distributing and rearranging, the given equation can be rewritten as $3x+bx=7b-21$. For most values of b, the only step remaining to solve the equation is to divide. For example, when $b=11$ we have $14x=56$, which gives $x=4$ upon dividing by 14. However, when $b=-3$ the x terms cancel, giving $0=-42$, which is not true for any value of x, so there is no solution when $b=\mathbf{-3}$.

4. We first demonstrate that there is a thrilling sequence of length six:

$$1,\ 2,\ -1,\ 28,\ -19,\ 100.$$

The sums of consecutive terms are 3, 1, 27, 9, and 81. To see that this is the shortest possible, suppose there were a three-term thrilling sequence of the form 1, a, 100. Then we would have $1+a=3^k$ and $a+100=3^l$ for integers $0\le k<l$. Subtracting these equations gives $99=3^l-3^k=3^k(3^{l-k}-1)$. Since 99 is divisible by 3^2 but no higher power of 3, we must have $k=2$, leading to $11=3^{l-k}-1$, or $3^{l-k}=12$, which doesn't work. Similar considerations rule out thrilling sequences of the form 1, a, b, 100; we would need $3^m-3^l+3^k=101$ in this case, which is not possible. (Proof?) Likewise a thrilling sequence of the form 1, a, b, c, 100 would imply $3^n-3^m+3^l-3^k=99$, which again cannot be done. Therefore the answer is indeed **6**.

5. Since M is the midpoint of $\overline{AB}$, we know that A is twice as high above $\overline{BC}$ as M, from which it follows that $area(ACN)=2area(MCN)$. However, there is a second method of comparing these two areas, using $\triangle PCN$, which is shaded in the diagram. We use the fact that the ratio of $area(ACN)$ to $area(PCN)$ is the same as the ratio of AN to PN. (Think of these segments as the bases of the two triangles to see why.) By the same token, the ratio of $area(PCN)$ to $area(MCN)$ is the same as the ratio of PC to MC. Letting $PM=x$, we have



$$\frac{area(ACN)}{area(MCN)}=\frac{area(ACN)}{area(PCN)}\cdot\frac{area(PCN)}{area(MCN)}=\frac{AN}{PN}\cdot\frac{PC}{MC}=\frac{10}{3}\cdot\frac{5}{x+5}.$$

We already know that this ratio equals 2, hence $6(x+5)=50$, which leads to $x=\mathbf{\frac{10}{3}}$. (*N.B.* There is also a short solution that employs the Theorem of Menelaus. The reader is invited to sort out the details.)

6. To obtain an arithmetic progression we must have

$$\begin{aligned}\log_8(7x+8)-\log_4(3x+4) &= \log_4(3x+4)-\log_2(x+2)\\ \Longrightarrow\quad \log_2(x+2)+\log_8(7x+8) &= 2\log_4(3x+4)\\ \Longrightarrow\quad \log_2(x+2)+\tfrac{1}{3}\log_2(7x+8) &= \log_2(3x+4)\\ \Longrightarrow\quad (x+2)^3(7x+8) &= (3x+4)^3\end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad 7x^4 + 23x^3 + 24x^2 + 8x &= 0 \\ \Longrightarrow \quad x(x+1)(7x^2 + 16x + 8) &= 0. \end{aligned}$$

(Some steps and explanations have been omitted.) The irrational solutions arise from the quadratic factor, which has roots $x = \frac{1}{7}(-8 + 2\sqrt{2})$ and $x = \frac{1}{7}(-8 - 2\sqrt{2})$. The latter is not a valid solution, however, since $7x + 8 < 0$ in this case so $\log_8(7x + 8)$ is not defined. We are left with $\mathbf{\frac{1}{7}(-8 + 2\sqrt{2})}$ as our solution.

7. The statement of the problem implies that $201^n \equiv 152^n \bmod 2009$. Multiplying through by 10^n gives $2010^n \equiv 1520^n \bmod 2009$, or just $1520^n \equiv 1 \bmod 2009$. Since $2009 = 41 \cdot 49$ this congruence is equivalent to having $1520^n \equiv 1 \bmod 41$ and $1520^n \equiv 1 \bmod 49$ separately. But fortunately $1520 \equiv 1 \bmod 49$, so the latter congruence always holds. We are left with the task of finding the smallest positive integer n for which $1520^n \equiv 1 \bmod 41$. Since $1520 \equiv 3 \bmod 41$ this reduces to $3^n \equiv 1 \bmod 41$. But observe that $3^4 \equiv 81 \equiv -1 \bmod 41$, and squaring both sides gives $3^8 \equiv 1 \bmod 41$. Thus $n = \mathbf{8}$ is the desired power.

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★ National Level ★

The Mandelbrot Competition

Round Five Solutions

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